Fast adaptive wavelet packets using interscale embedding of decomposition structures

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A B S T R A C T

The most widely used basis selection algorithm for adaptive wavelet packets is the optimal basis selection method, which first grows a full wavelet packet tree and then prunes it into the optimal tree that gives the minimum cost. We observed that there exists the interscale embedding property in wavelet packet decomposition structures between interscale subbands along the same orientation. Based on this observation, we propose a fast basis selection algorithm by first decomposing a dyadic wavelet subband with the decomposition structure of its parent subband, before applying the optimal basis selection method for further decomposition. Experiments show that the proposed algorithm generates almost the same wavelet packet decomposition structures as the optimal basis selection method while significantly reducing the computational complexity for some image classes.

1. Introduction

Wavelet packets form a library of bases with dyadic wavelets as its members (Coifman and Meyer, 1990). By further decomposing dyadic wavelet subbands, wavelet packets have stronger capability in capturing localized high-frequency components. Moreover, wavelet packets can adapt to signal characteristics by selecting a “best” wavelet packet basis in the library (Coifman and Wickerhauser, 1992; Ramchandran and Vetterli, 1993). As a result, wavelet packets show better performance than dyadic wavelets in many applications.

The essential step of image processing using adaptive wavelet packets is to select the best basis to facilitate subsequent processing procedures. Given a cost function, one can find the best wavelet packet by exhaustive search over the whole wavelet packet library. However, it is not practical due to its high computational complexity. Using an additive cost function, Coifman and Wickerhauser (1992) proposed an optimal basis selection algorithm based on the Bellman’s principle of optimality in dynamic programming. So far, most adaptive wavelet packets use this algorithm for basis selection (Ramchandran and Vetterli, 1993; Yang et al., 2008; Hong et al., 2001; Meyer et al., 2000; Rajpoot et al., 2003; Xiong et al., 1998). Although the computational complexity is much more lower than the exhaustive search ($N \log_2(N)$ vs. $N^2\log_2N$, where $N$ is the signal length), the computation still significantly increases with the maximum decomposition level due to the exponential growth of the number of possible decomposition structures.

Besides the conventional (isotropic) decomposition for wavelet packets, the anisotropic decomposition has been advocated to enhance the capability of wavelet packet in capturing edges and contours that have anisotropic supports (Xu and Do, 2003; Yang et al., 2008). In each level of the isotropic decomposition, the separable row decomposition and column decomposition are subsequently applied; in each level of the anisotropic decomposition, only the row or column decomposition is performed. As a result, including isotropic wavelet packets as special cases, the anisotropic wavelet packets greatly enlarge the wavelet packet library, and thus provide more efficient image representation. However, the number of decomposition levels of the anisotropic decomposition is doubled compared with that of the isotropic decomposition, significantly increasing the required computation for the optimal basis selection.

In this letter, we propose a fast basis selection algorithm based on the observation that the decomposition structure of a dyadic wavelet subband is embedded in its child subband. The main idea is to first decompose a dyadic wavelet subband with the decomposition structure of its parent subband before applying the optimal basis selection method. Experimental results show that the proposed algorithm is up to 20 times faster than the optimal basis selection, only slightly compromising the cost. Image denoising and coding results verify the effectiveness of the proposed algorithm.

2. The optimal basis selection for anisotropic wavelet packets

The optimal basis selection algorithm for anisotropic wavelet packets (Yang et al., 2008) is extended from that for conventional...
isotropic wavelet packets in (Coifman and Wickerhauser, 1992). Given a predefined maximum decomposition level, an image is first decomposed into a full wavelet packet, where all wavelet packet subbands have the same size and hence the same frequency resolution. Fig. 1(a) shows the full wavelet packet decomposition of Barbara. As shown in Fig. 2(a), the decomposition path can be organized into a quadtree, in which each non-leaf node has four children: two of them are obtained by row decomposition and the other two are generated from column decomposition. Due to the difference of decomposition, the quadtree of anisotropic wavelet packets (anisotropic quadtree) is different from that of isotropic wavelet packets (isotropic quadtree). In isotropic decomposition, a subband can be either kept as is or decomposed into four subbands. Accordingly, in an isotropic quadtree, a non-leaf node can have no child or four children. However, in anisotropic decomposition, a subband can be either kept as is or decomposed into two subbands by row decomposition or column decomposition. Accordingly, in a pruned anisotropic quadtree, a non-leaf node can have no child or two children (generated by row decomposition or column decomposition). As shown in Fig. 2(b) and (c), unlike in the isotropic tree, the admissible trees of an anisotropic quadtree are binary trees. When an image is decomposed to a given finest frequency resolution for both the vertical and horizontal, the depth of an anisotropic quadtree is twice as that of an isotropic quadtree, which significantly increases the required computation for basis selection.

Using an additive cost function, the optimal anisotropic wavelet packet with the minimum cost can be selected by pruning the full packet tree from bottom to up. For each non-leaf node, the one that gives the minimum cost is selected among the three possible decomposition decisions: no decomposition, row decomposition, and column decomposition; part of its offspring are pruned out according to the selected decision. The best admissible tree is found when the pruning procedure reaches the root node. For example, Fig. 2(c) shows an optimal pruned tree of the quadtree in Fig. 2(a) in terms of the $\ell_1$ norm of resulting coefficients. See (Yang et al., 2008) for a detailed description of this algorithm.

Listed below are several cost functions in the literature (Coifman and Wickerhauser, 1992; Hong et al., 2001; Xiong et al., 1998):

- Entropy function: $\sum_{i,j} x_{ij}^2 \log(x_{ij}^2)$;
- $\ell_1$ norm: $\sum_{i} |x_{i}|$;
- Log energy: $\sum_{i} \log(x_{i}^2)$, where let log(0) be zero;
- Threshold: $\#\{i : |x_{i}| > T\}$, where $T$ is a threshold;
- Rate distortion: $D(x, Q) + iR(x, Q)$, where $D(x, Q)$ and $R(x, Q)$ are the distortion and bit rate of quantized coefficients with a step size $Q$, respectively.

It has been verified that the entropy function does not have connection with the actual entropy of wavelet coefficients, and usually cannot find any meaningful basis (Meyer et al., 2000). The threshold cost and the rate-distortion cost require additional parameters. Determining proper parameters is another optimization problem, which highly depends on specific applications. Despite the simplicity, the $\ell_1$ norm and log energy are quite effective in selecting useful bases (Hong et al., 2001). Moreover, Donoho (2006) showed that minimizing the $\ell_1$ norm can obtain the sparsest representation over the solution space, which is quite useful in many image

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1 As shown in Fig. 2(a), the nodes at the second deepest level of the quadtree only have two children since they have reached the maximum decomposition level on one direction.
processing tasks. Therefore, for generality, we use the $\ell_1$ norm and log energy for the reported results. Fig. 1(b) and (c) presents the optimal anisotropic decomposition structures in terms of the $\ell_1$ norm and log energy, respectively.

3. Fast basis selection using interscale embedding

3.1. Interscale embedding of decomposition structures

Wavelet subbands of images exhibit correlations across scales along the same spatial orientation. Exploiting the interscale correlations of wavelet subbands has become an important tool in image processing such as denoising (Sendur and Selesnick, 2002) and coding (Shapiro, 2003). For a wavelet subband, the subband at the coarser scale along the same spatial orientation is referred to as its parent subband while the subband at the finer scale is referred to as its child subband. We observed that the interscale correlation also exists in decomposition structures of adaptive wavelet packets: with a high probability, the selected decomposition structure for a subband is embedded in its child subband, which we called the interscale embedding property of decomposition structures.

Taking the wavelet packet in Fig. 1(b) as an example, quadtreess of decomposition structures on subband $HL_2$, $HL_1$, and $HL_0$ are shown in Fig. 3(a), (b), and (c), respectively. It can be observed that, the quadtree associated with a wavelet subband is part of the quadtree associated with its child subband: the quadtree of subband $HL_2$ (Fig. 3(a)) is part of the quadtree of subband $HL_1$ (Fig. 3(b)) while the quadtree of subband $HL_1$ (Fig. 3(b)) is part of the quadtree of subband $HL_0$ (Fig. 3(c)). That is, the decomposition structure of subband $HL_2$ is embedded in that of subband $HL_1$, and the decomposition structure of subband $HL_1$ is embedded in that of subband $HL_0$. The interscale embedding of decomposition structures also holds for interscale subband pairs along other orientations, i.e., $LH$ subbands and $HH$ subbands (except for the $HH_1$–$HH_0$ pair), and holds for the wavelet packet in Fig. 1(c).

We further justify the above observation with more evidences. Six test images are shown in Fig. 4: three of them are photos of natural scenes; Fingerprint presents local details of finger skin; Pentagon is an aerial picture of the Pentagon building; Brain is an MRI image. The test images are four-level decomposed by DWT followed by adaptive anisotropic wavelet packet decomposition. As shown in Table 1, 94 out of 108 interscale subband pairs present the interscale embedding, which suggests that the interscale embedding of wavelet packet decomposition structures is general for various kinds of images.

3.2. Fast basis selection

Using the interscale embedding as a prior, we develop a fast basis selection algorithm by initially decomposing a dyadic wavelet subband using the decomposition structure of its parent subband.
Since the frequency resolution of a subband is lower than its parent subband, each resulting subband after the initial decomposition does not reach the finest allowable frequency resolution (associated with the full packet). We further decompose these subbands with the optimal basis selection algorithm up to the finest allowable frequency resolution. Let an image be decomposed by 2-D DWT up to $J$ levels, producing one low-pass subband and 3J high-pass subbands $S_{i,j}$, where $i \in \{ LH, HL, HH \}$ and $j \in \{ 0, 1, \ldots, J-1 \}$. Subband $S_{j,1}$ has the finest frequency resolution while subband $S_{0,1}$ has the coarsest frequency resolution. The finest allowable frequency resolution of anisotropic wavelet packets along both directions is limited to $2^{-j}$ on the 2-D frequency plane (normalized to [01] x [01]). In other words, the decomposition is allowed only when the width (height) of the subband is larger than that of the low-pass subband. Let $T_{n}$ be the tree representation of the selected decomposition structure for subband $S_{i,j}$. Since subband $S_{j,1}$ already has the finest frequency resolution and need not to be further decomposed, $T_{j,1}$ is initialized to a quadtree with only a root node. The proposed fast basis selection algorithm is described in Algorithm 1.

The computation of the optimal basis selection increases with the maximum depth of the full wavelet packet tree. By using the interscale embedding as a prior, subbands are first decomposed into smaller subbands with decomposition structures of their parent subbands before applying the optimal basis selection algorithm. Therefore, the required computation of the proposed algorithm is reduced. Taking subband $HL_1$ in Fig. 1(b) as an example (see also Fig. 3(b)), in stead of growing a full wavelet packet tree, our method first decomposes subband $HL_1$ using the decomposition structure of $HL_0$, i.e., one-level row decomposition, and then selects optimal decomposition structures for the resulting two subbands. Therefore, we do not perform the row decomposition and its subsequent possible decompositions, nor the optimality of these decomposition structures is checked. For subband $HL_1$, at least half of computational cost can be saved compared with the optimal basis selection method. Qualitatively, the more the subbands at coarser scales are decomposed, the less computation is required since subbands at finer scales are decomposed into smaller subbands at the initial decomposition stage. A theoretical analysis on the reduction of computation is difficult since the decomposition structure for each subband is signal-dependent.

As the top-down searching strategy by Taswell (1996), the proposed algorithm slightly increases coefficient costs compared with the optimal basis selection method, and is thus suboptimal. However, the increase of costs does not affect the performance in practical applications, which is shown in the next section.

### Table 1

Relationship of interscale subband pairs for six test images. Y: the decomposition structures of the subband pair show the interscale embedding; N: the complement of Y.

<table>
<thead>
<tr>
<th>Subband pair</th>
<th>Barbara</th>
<th>Fingerprint</th>
<th>Boat</th>
<th>Goldhill</th>
<th>Pentagon</th>
<th>Brain</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i_{1}$ Norm</td>
<td>LH_{i_{1}}</td>
<td>LH_{i_{1}}</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>$i_{2}$ Norm</td>
<td>LH_{i_{2}}</td>
<td>LH_{i_{1}}</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>$i_{3}$ Norm</td>
<td>LH_{i_{2}}</td>
<td>LH_{i_{1}}</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>$i_{4}$ Norm</td>
<td>LH_{i_{2}}</td>
<td>LH_{i_{1}}</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>$i_{5}$ Norm</td>
<td>LH_{i_{2}}</td>
<td>LH_{i_{1}}</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>$i_{6}$ Norm</td>
<td>LH_{i_{2}}</td>
<td>LH_{i_{1}}</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
</tbody>
</table>

### Algorithm 1. Fast basis selection algorithm

**Initialize:** Set $T_{j,1}$ to a quadtree with a root node, $i \in \{ LH, HL, HH \}$

**for** $j = 2$ to $0$ **do**

**for** $i \in \{ LH, HL, HH \}$ **do**

Decompose subband $S_{i,j}$ with the decomposition structure $T_{j+1}$ of the parent subband $S_{i,j+1}$.

Further decompose the resulting wavelet packet subbands with the optimal basis selection and obtain $T_{i,j}$.

**end for**

**end for**

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### 4. Experimental results

In the following experiments, the CDF 9/7 filters are employed in both wavelet transform and wavelet packet decomposition. Test images are five-level decomposed for both DWT and wavelet packets. Experiments are carried out on a desktop with an Intel Pentium (R) IV 3.0-GHz CPU and 1.5-GB RAM.

#### 4.1. Running time and cost

For comparison, running time of adaptive wavelet packets selected by both the optimal basis selection method and the proposed method are tabulated in Table 2 while coefficient costs are shown in Table 3. Algorithms using the log energy cost consume about twice as much running times as those using the $i_{1}$ norm cost, because the calculation of the log energy is more time-demanding than that of the $i_{1}$ norm cost and is frequently called during building the full wavelet packets. As shown in Table 2, the proposed method is about 2–20 times faster than the optimal basis selection algorithm. Also, the improvement is more significant for images with rich oscillatory features, e.g., Barbara and Fingerprint. This is because subbands at coarser scales are decomposed with more levels for these images, and hence are decomposed into smaller subbands before applying the optimal basis selection algorithm.

For some cases, the proposed method produces the same solutions, hence the same costs, as the optimal method. For images whose subband pairs do not always obey the interscale embedding, e.g., Barbara and Fingerprint, the selected structures result in slight increase of the coefficient cost. As shown in Table 3, the increase of the $i_{1}$ norm cost is marginal (within 0.3%) for all the test images. The log energy cost function is negative when the absolute values of coefficients are smaller than one in a geometric mean sense, and has a sharp slope when coefficients approach zero. Most

<table>
<thead>
<tr>
<th>Image</th>
<th>$i_{1}$ Norm</th>
<th>Log energy</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Optimal</td>
<td>Proposed</td>
</tr>
<tr>
<td>Barbara</td>
<td>3.81</td>
<td>0.74</td>
</tr>
<tr>
<td>Fingerprint</td>
<td>3.82</td>
<td>0.25</td>
</tr>
<tr>
<td>Boat</td>
<td>3.81</td>
<td>1.82</td>
</tr>
<tr>
<td>Goldhill</td>
<td>3.82</td>
<td>1.88</td>
</tr>
<tr>
<td>Pentagon</td>
<td>4.17</td>
<td>2.28</td>
</tr>
<tr>
<td>Brain</td>
<td>4.20</td>
<td>1.13</td>
</tr>
</tbody>
</table>
wavelet coefficients are very small, especially for images with low complexities such as MRI images. Therefore, the increase of the log energy cost is relatively more significant (within 2.0%). Note that the values of log energy costs for the MRI image Brain are negative since most its wavelet coefficients are nearly zeros.

4.2. Image denoising and coding results

We also present image denoising and coding results of adaptive anisotropic wavelet packets chosen by the proposed basis selection algorithm for Barbara and Fingerprint, whose selected wavelet packets are different from those by the optimal method. For denoising the wavelet coefficients, we use the BayesShrink method (Chang et al., 2000): \[ \hat{x}_i = \text{soft}(x_i, \sigma_i^2 / \sigma_x^2), \] where \( \text{soft}(x, T) = \text{sign}(x) \cdot \max(|x| - T, 0) \) is the soft thresholding function. The standard deviation of noises \( \sigma_i \) is estimated by a median estimator (Donoho and Johnstone, 1994), and that of the underlying clean coefficients \( \sigma_x \) in a subband \( S \) is estimated as: \[ \hat{\sigma}_S = \left( \sum_{x_i \in S} x_i^2 / M \right)^{0.5} / \sigma_x, \] where \( M \) is the number of coefficients in \( S \). As shown in Table 4, the adaptive wavelet packets consistently outperform DWT by up to 0.8 dB. The proposed algorithm even gives slightly better performance than the optimal basis selection. The reason is that, in the presence of noise, the proposed algorithm is more reliable due to the prediction of decomposition structures from parent subbands, in which coefficients have higher signal-to-noise ratio (SNR). In all, the basis selection algorithms with the log energy cost gives slightly better denoising results than those with the \( \ell_1 \) norm cost.

For image coding, wavelet coefficients are coded with a TCE coder (Tian and Hemami, 2004), which produces scalable bit streams. As shown in Table 5, the proposed basis selection algorithm gives almost the same coding results as the optimal one for both cost functions, and the wavelet packet coders significantly outperform JPEG2000. We also compare the proposed algorithm with an image coding algorithm using fast wavelet packets (FWP) (Meyer et al., 2000). In FWP, a fast implementation of the optimal basis selection method (Coifman and Wickerhauser, 1992) is used; the cost function is designed to take into account the cost of entropy coding which is also optimized in a context-based manner. Despite the simplicity, the proposed algorithm shows on average a 0.3 dB gain over the FWP method for Barbara. For Fingerprint, the results of FWP are missing since the test image in (Meyer et al., 2000) is not the same as the one used in our work, and is not available on several popular image databases. In addition, the scalable coder is competitive to the SFQ on wavelet packet (Xiong et al., 1998), in which both the decomposition structure and quantizer are optimized for each rate.

The motivation of the proposed method is to accelerate the anisotropic wavelet packets. The interscale embedding property is also observed in isotropic wavelet packets, which are special cases of anisotropic wavelet packets. Therefore, the proposed fast basis selection method can also be used in isotropic wavelet packets. Since the decomposition level of isotropic decomposition up to the full packet is fewer than that of the anisotropic decomposition, the save of running time for isotropic wavelet packets is not so significant.

5. Conclusion

This letter proposes a fast basis selection using the interscale embedding property of decomposition structures. The proposed method is based on the observation that the decomposition structures of adaptive wavelet packets present interscale embedding: the selected decomposition structure for a subband is embedded in that of the child subband. Therefore, a subband can be first decomposed with the decomposition structure of its parent subband before applying the optimal basis selection algorithm for further decomposition. Experimental results show that the proposed algorithm significantly reduces the required computation while selecting almost the same decomposition structures as the optimal basis selection algorithm for some image classes. The effectiveness of the proposed algorithm is verified in image denoising and coding.

In this letter, we investigate the interscale prediction on the optimal basis selection algorithm with the \( \ell_1 \) norm cost and log...
energy cost. It can be considered as a principle to achieve fast basis selection, and can be incorporated with other basis selection algorithms and cost functions. For example, using a rate-distortion cost, the interscale embedding property can be used to reduce the search space in the joint optimization (Xiong et al., 1998) of decomposition structures and quantizers for image coding using adaptive wavelet packets.

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